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On the degree of the minimal equation of the matrices in first-order relativistic wave equations

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Abstract. Starting with an analysis of the general structure of the matrix β^0 entering in first-order relativistic wave equations, we show that the degree of the minimal equation of β^0 is determined by the size and nature of the various spin blocks of the 'skeleton matrix' of the theory. Since it is the numbers of Lorentz irreducible representations contributing to particular spins which determine the sizes of the spin blocks (and the value j_m of the maximum spin contained in ψ has no direct bearing on these), the reason for the failure of the Umezawa–Visconti rule and its extrapolation by Chandrasekaran *et al* becomes clear. We obtain some general results concerning the minimal degree in certain types of theories and on certain procedures whereby the minimal degree can be raised without altering j_m, and finally analyse a few interesting examples.

1. Introduction

It has long been known, from the work of Harish-Chandra (1947), that manifestly covariant relativistic wave equations of the first-order form

$$(-i\beta^\mu\partial_\mu + m)\psi = 0 \tag{1.1}$$

describe particles of unique mass m if and only if the minimal equation of β^0 (and of the β^i) is of the form

$$(\beta^0)^{l+2} = (\beta^0)^l \tag{1.2}$$

where l is some non-negative integer. By an ingenious argument based on the fact that products of r factors of β^μ transform like tensors of rank r under similarity transformation by the matrices $S(\Lambda)$ constituting the representation of the Lorentz group (LG) according to which ψ transforms, Umezawa and Visconti (1956) (see also Takahashi 1969) concluded that the degree $l+2$ of the minimal equation cannot exceed $(2j_m+1)$ where j_m is the maximum spin involved in the representation $S(\Lambda)$. However, a flaw in the argument has recently been pointed out by Glass (1971) who has also constructed a counter example to the Umezawa–Visconti theorem: a theory with $j_m=\frac{3}{2}$ but $l+2=5>(2j_m+1)$. Nevertheless the deeper reasons for the inapplicability of the Umezawa–Visconti argument have not been analysed, nor have the factors which are in fact responsible for any limitations on the value of l been identified. In this situation there has been, perhaps not surprisingly, a continuing misunderstanding of the relevance of the transformation property of products of the β^μ, referred to above, to the

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question of the minimal degree, $l+2$. A case in point is the claim of Chandrasekaran *et al*† (1972) (based on a rather 'fuzzy' though plausible-sounding interpretation of the transformation property), which is cited in the recent work of Cox (1978), namely that a lower bound on l is imposed by the physical spin s ($s \leqslant j_m$) of the particle described by the wave equation (1.1). No such bound in fact exists; the Hurley equation (Hurley 1971) for arbitrary spin s, which has $l = 1$ independent of s, provides a counter example.

Our aim in this paper is to elucidate the problem which is of particular interest in view of recent work (Mathews *et al* 1979, Seetharaman *et al* 1979, Amar and Dozzio 1972, 1975) showing that the number and variety of consistency problems which arise in interacting higher spin theories are strongly correlated with the degree $l+2$ of the minimal equation of β^0. We show that l is determined primarily by the dimension and structure of the 'skeleton matrix' of the β^μ (for definition see § 2). The highest spin j_m occurring in $S(\Lambda)$ enters only indirectly, through the number of irreducible representations (hereafter abbreviated to IR's) with which the IR's containing j_m can be coupled through the β^μ.

In § 2 we focus briefly on the salient features of the general structure of the β^0 matrix which are relevant to our work—features which follow from its behaviour as the time-like component of a four-vector.

2. Structure of β^0

The finite dimensional IR's of the proper LG are conveniently labelled by (m, n), associated with the eigenvalues $m(m+1)$ and $n(n+1)$ of the Casimir operators $\boldsymbol{M}^2$ and $\boldsymbol{N}^2$ where $\boldsymbol{M}, \boldsymbol{N}$ are given in terms of the generators $\boldsymbol{J}$ and $\boldsymbol{K}$ of rotations and boosts by

$$\boldsymbol{M} = \tfrac{1}{2}(\boldsymbol{J} + \mathrm{i}\boldsymbol{K}) \qquad \boldsymbol{N} = \tfrac{1}{2}(\boldsymbol{J} - \mathrm{i}\boldsymbol{K}). \tag{2.1}$$

The abbreviations $\tau \equiv (m, n)$, $\tau' \equiv (m', n')$, etc., are often used. The basis states within an IR may be chosen to diagonalise either M_3 and N_3 or $\boldsymbol{J}^2$ and J_3. The latter basis is best suited for the analysis of the spin content, and we shall adopt it in this paper (Gel'fand *et al* 1963, Hurley and Sudarshan 1974). Note that since $\boldsymbol{J} = \boldsymbol{M} + \boldsymbol{N}$ (and $\boldsymbol{M}$ and $\boldsymbol{N}$ are mutually commuting angular momentum-like operators), states with $j = (m+n), (m+n-1), \ldots, |m-n|$ are present in the IR space (m, n).

As shown by Bhabha (1945) (see also Corson 1953) the β^μ can have non-vanishing matrix elements between two IR's only if $|m'-m| = |n'-n| = \frac{1}{2}$. Further, since β^0 commutes with $\boldsymbol{J}$ it can have no matrix elements connecting states with unequal values of j; and the $(2j+1) \times (2j+1)$ block connecting the $(2j+1)$ states with given j in τ with similar states in τ' must be a multiple of the unit matrix. This multiple decomposes into one factor, say $g_j^{(\tau',\tau)}$, which is in the nature of a Lorentz group Clebsch–Gordan coefficient‡ (determined purely by the fact that β^0 is the time-like component of a four-vector), and another factor $c^{(\tau',\tau)}$ which is a 'reduced matrix element' characterising the *particular* four-vector in question. This second factor, naturally, depends only on the Casimir operators of the IR's connected, and not on the internal quantum numbers (j, σ) of any IR. Thus the part of β^0 connecting states with angular momentum j in τ' with those in τ is

$$c^{(\tau',\tau)} g_j^{(\tau',\tau)} I_j \tag{2.2}$$

† See also Santhanam and Tekumalla (1974).

‡ Values of $g_j^{(\tau',\tau)}$ for specific τ', τ can be easily worked out. See, for instance, Hurley and Sudarshan (1974).

where I_j is the unit matrix of dimension $(2j+1)$. We shall refer to this as the (τ', τ) part of the spin-j block of β^0. If either τ' or τ (or both) occurs in $S(\Lambda)$ more than once, then the value of the reduced matrix element connecting each pair (one of the τ' with one of the τ) is unrelated to that for any of the other pairs, but all are multiplied by the same $g_j^{(\tau',\tau)}$. Then instead of (2.2) one has

$$C^{(\tau',\tau)} g_j^{(\tau',\tau)} \times I_j \tag{2.3}$$

where $C^{(\tau',\tau)}$ is an $\alpha' \times \alpha$ matrix with arbitrary elements (α and α' being the multiplicities of τ and τ' in $S(\Lambda)$). The direct product with I_j indicates the fact that in the spin-j block, each element of $C^{(\tau',\tau)}$ appears multiplied by the unit matrix I_j of dimension $(2j+1)$. The spin-j block is made up as an array of sub-blocks (2.3) associated with all those pairs τ, τ' such that $\tau, \tau' \in S_j(\Lambda)$. Here $S_j(\Lambda)$ is the direct sum of all those IR's in $S(\Lambda)$ which contain the particular j of interest, i.e. such that $(m+n) \geqslant j \geqslant |m-n|$. (Note that the $S_{(j)}(\Lambda)$ for different j values are in general overlapping, not exclusive.) We shall denote this spin-j block by $\beta^0_{(j)}$. We shall also introduce the name *skeleton matrix* or reduced matrix for the matrix C made up of the blocks $C^{(\tau',\tau)}$ for all $\tau', \tau \in S(\Lambda)$. It is of dimension $\Sigma\alpha$, equal to the total number of IR's (of the proper LG) which are present in $S(\Lambda)$, including all multiplicities. The arrangement of blocks in the skeleton matrix depends of course on the order in which the various IR's are taken.

A type of ordering which leads to considerable simplicity in the form of C is the following. Classify all the IR's occurring in $S(\Lambda)$ into two sets, such that if both τ and τ' are within one and the same set, $(m-m')$ and $(n-n')$ are integral, while if τ belongs to one set and τ' to the other, then $(m-m')$ and $(n-n')$ are both half-odd-integral. In labelling the rows and columns of C, exhaust all the IR's of either one of the above sets and then proceed to the IR's of the other set. In so doing we obtain for β^0 the form

$$\beta^0 = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}. \tag{2.4}$$

This must evidently be the case since β^0 can have non-vanishing elements only between IR's τ, τ' such that $|m'-m| = |n'-n| = \frac{1}{2}$, a condition which cannot be met if τ and τ' belong to one and the same set as defined above.

It may be noted that each of the spin blocks $\beta^0_{(j)}$ will also then necessarily have a form similar to the above.

3. The spin blocks $\beta^0_{(j)}$ and the minimal degree of β^0

When a basis which diagonalises $\boldsymbol{J}^2$ is chosen and β^0 is expressed as a direct sum of its spin-j blocks (i.e. with block $\beta^0_{(j)}$ along the diagonal and all off-diagonal blocks vanishing), the Harish-Chandra condition (1.2) can evidently be translated into conditions on the spin blocks. One is thus led to the requirement that

$$(\beta^0_{(j)})^{l_j+2} = (\beta^0_{(j)})^{l_j} \tag{3.1}$$

for one or more values of j, with $\beta^0_{(j)}$ nilpotent for all other j. If it is further stipulated that only a single spin (say s) be admitted by (1.1), it is necessary that the minimal equations of the spin blocks be

$$(\beta^0_{(j)})^{l_j+2}\delta_{js} = (\beta^0_{(j)})^{l_j}. \tag{3.2}$$

If the $\beta^0_{(j)}$ are imagined to be reduced to Jordan canonical form by suitable similarity transformations, (3.2) tells us that the reduced form of $\beta^0_{(j)}$ contains at least one nilpotent irreducible Jordan block of dimension l_j (besides possibly others of lower dimension but none bigger); for $j = s$ alone there is an additional strictly diagonal part made up of eigenvalues $+1$ and -1. In the following l_j will be referred to as the *degree of nilpotency of* $\beta^0_{(j)}$. A little reflection now shows that $(\beta^0)^{l+2} = (\beta^0)^l$ if l is the largest of the l_j, but not for any lower value of l. In other words, l in the Harish-Chandra equation is equal to the largest of the l_j, i.e. the dimension of the largest nilpoint irreducible Jordan block in the canonical form of β^0. The question of the minimal degree of β^0 thus reduces to the problem of finding or placing bounds on the dimensions of nilpotent blocks in β^0, given the IR's present (and their multiplicities) in $S(\Lambda)$.

At this point it is self-evident that there need not be any immediate relation between l and the spin content of $S(\Lambda)$. We have seen that $\beta^0_{(j)}$ is of the form $(Cg)_{(j)} \times I_j$ where $(Cg)_{(j)}$ is made up of the blocks $C^{(\tau',\tau)}g_j^{(\tau',\tau)}$ of (2.3). The dimension of I_j is $(2j+1)$, but this fact obviously has no bearing on the value of l_j; it is the matrix $(Cg)_{(j)}$ which determines the value of l_j. The dimension of $(Cg)_{(j)}$, i.e. the number of IR's contributing to spin j in $S(\Lambda)$—counting all multiplicities—places an upper bound on l_j.

This last mentioned fact leads immediately to a negation of the claim of Chandrasekaran *et al* (1972) that l cannot be lower than $(2s+1)$ where s is the unique spin allowed by the wave equation. The example of the Hurley equation brings home this point forcefully. The equation involves just two IR's $(s, 0)$ and $(s-\frac{1}{2}, \frac{1}{2})$ each occurring once. Only the latter contributes to spin $(s-1)$ while both the IR's contribute to spin s. Thus $(Cg)_{(s)}$ is a 2×2 matrix whose eigenvalues must be $+1$ and -1 while $(Cg)_{(s-1)} = 0$. Hence $l_s = 0$ and $l_{(s-1)} = 1$, giving $l = 1$ i.e. $(\beta^0)^3 = \beta^0$. The degree of the minimal equation is therefore *less* than $(2s+1)$ for any $s > 1$. The fact that there is no naturally-defined parity operator or a conventional kind of Lagrangian in the Hurley theory is not really of relevance, insofar as the Umezawa–Visconti proof and Chandrasekaran *et al*'s extrapolation of it do not rest on the existence of either. In any case, by extending the wavefunctions to include also parts $(0, s)$ and $(\frac{1}{2}, s-\frac{1}{2})$ conjugate to what was considered above, one can readily construct a Lagrangian theory with parity invariance† without raising the value of l.

The failure of the Umezawa–Visconti upper bound on l can also be understood in a similar fashion. In the Glass equation, which involves the six IR's $(1, \frac{1}{2}) + 2(0, \frac{1}{2}) + (\frac{1}{2}, 1) + 2(\frac{1}{2}, 0)$, only two of the IR's contribute to the spin-$\frac{3}{2}$ block leading to $l_{3/2} = 0$ as it is required that the equation should describe spin $\frac{3}{2}$; all six contribute to $j = \frac{1}{2}$ but the degree of nilpotency of this block turns out to be $l_{1/2} = 3$. Consequently, the minimal degree of β^0 is 5, exceeding the Umezawa–Visconti value 4 (corresponding to $j_m = \frac{3}{2}$).

4. Some general results

4.1. Case of half-integer spins with parity invariance

If invariance under space inversion is required, it is necessary that every IR $\tau \equiv (m, n)$ present in $S(\Lambda)$ must be accompanied by its conjugate $\dot{\tau} = (n, m)$. It is also readily seen that when classification of the IR's into two sets is done as described at the end of the last section, $\dot{\tau}$ is in the second set if τ is in the first, and vice versa. Further, by ordering the IR's as $\tau_1, \tau_2, \ldots, \dot{\tau}_1, \dot{\tau}_2, \ldots$, and using the fact that $C^{(\tau',\tau)} = C^{(\dot{\tau}',\dot{\tau})}$ for space-inversion

† This theory will give two spin-s particles degenerate in mass, but once again this fact is of no relevance to the treatments of Umezawa and Visconti and Chandrasekaran *et al.*

invariance, one gets the form (2.4) for β^0, with $A=B$. This matrix is equivalent to $\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$, and correspondingly, $\beta^0_{(j)}$ has an equivalent form

$$\begin{pmatrix} A_{(j)} & 0 \\ 0 & -A_{(j)} \end{pmatrix}.$$

It is of course the direct product of a unit matrix of dimension $(2j+1)$ with a skeleton part whose dimension, say L_j, is equal to the number of IR's in $S(\Lambda)$ which contribute to spin j. For $j \neq s$, $\beta^0_{(j)}$ and hence $A_{(j)}$ are to be nilpotent and it is clear that the degree of nilpotency l_j cannot exceed the dimension $\frac{1}{2}L_j$ of the skeleton part of $A_{(j)}$. Thus

$$l_j \leq \tfrac{1}{2}L_j \qquad (j \neq s).$$

For $j=s$, a single unit eigenvalue must come from the skeleton part of $A_{(j)}$, assuming there is no mass degeneracy. So the degree of nilpotency must obey

$$l_s \leq (\tfrac{1}{2}L_s - 1).$$

These relations determine the upper bound on l, once $S(\Lambda)$ is specified. We have already seen from the example of the Glass equation that by the device of increasing the multiplicities of IR's (without introducing any more distinct IR's) one can increase the l_j and hence the minimal degree $l+2$, without increasing the maximal spin j_m contained in $S(\Lambda)$. Another systematic way of altering a given equation so as to increase the minimal degree is through the introduction of barnacles.

4.2. Increase of minimal degree through barnacles

If all the elements of one or more rows (or columns) of the skeleton matrix vanish, then it can be shown that the parts of the wavefunction corresponding to these rows/columns either vanish or can be expressed in terms of the rest of the wavefunction. Such parts which have no essential or independent roles are called barnacles (Hurley and Sudarshan 1975, Khalil 1978).

Given a β^0 in the form (2.4), such that $l=2j_m-1$, we can readily obtain another β^0 with a higher minimal degree (without raising j_m) by introducing a suitable barnacle. For example, suppose the original β^0 was (2.4), and the augmented β^0—call it $\beta^{0\prime}$—has

$$A' = \begin{pmatrix} A \\ A_a \\ 0 \end{pmatrix} \qquad \text{and} \qquad B' = (B \; 0 \; B_a) \tag{4.1}$$

instead of A and B respectively (with the null diagonal blocks in β^0 also correspondingly enlarged). Suppose also that the minimal nilpotent degree l of the original β^0 was an even number, $2r$. Its minimal equation, $(\beta^0)^{2r+2}=(\beta^0)^{2r}$, then yields

$$\begin{aligned} (AB)^{r+1} &= (AB)^r, \\ (BA)^{r+1} &= (BA)^r, \end{aligned} \tag{4.2}$$

r being the smallest integer for which these properties hold. These equations do not in general force corresponding equations (with the same value of r) on $\beta^{0\prime}$, for $(A'B')^{r+1}$, which is equal to

$$\begin{pmatrix} (AB)^{r+1} & 0 & (AB)^r AB_a \\ A_a B(AB)^r & 0 & A_a(BA)^r B_a \\ 0 & 0 & 0 \end{pmatrix},$$

does not in general reduce to $(A'B')^r$ by virtue of (4.2) alone. However, one can readily verify that $(\beta_0')^{2r+3}$ is necessarily equal to $(\beta_0')^{2r+1}$. So the minimal nilpotent degree of β_0' is, in general, $l' = 2r+1 = l+1$.

One can repeat the process, starting now with β_0', to produce a β_0'' whose minimal degree is still higher, and so on. If the intention is to construct an example violating the Umezawa–Visconti condition, it would be necessary that the additional parts to the wavefunction, which are introduced in the form of barnacles, do not contain any spin higher than the original j_m.

It can be shown that by choosing A_a and B_a appropriately, parity invariance can be preserved even while adding barnacles (Khalil 1978).

5. Examples

5.1. Augmented Duffin–Kemmer β^0

Suppose that the Duffin–Kemmer wavefunction (for spin-1 particles) for which $S(\Lambda) \sim (1,0) + (\frac{1}{2}, \frac{1}{2}) + (0,1)$ is enlarged to include a scalar part $(0,0)$ as a barnacle. The most general form for $\beta^{0\prime}$ is shown below:

τ' \ τ	j' \ j	$(\frac{1}{2}, \frac{1}{2})$ 0	$(\frac{1}{2}, \frac{1}{2})$ 1	$(1,0)$ 1	$(0,1)$ 1	$(0,0)$ 0
$(\frac{1}{2}, \frac{1}{2})$	0			0	0	γ
	1			α	β	0
$(1,0)$	1	0	α'			
$(0,1)$	1	0	β'			
$(0,0)$	0	γ'	0			

The two diagonal blocks which have been left blank are null. Each element in the above is a $(2j'+1) \times (2j+1)$ matrix; when $j = j'$ it is a multiple of the unit matrix of dimension $(2j+1)$. The Duffin–Kemmer β^0 is obtained if the first row and column of the above are deleted and the usual values (which are consistent with (5.2) below) are assigned for $\alpha, \beta, \alpha', \beta'$.

The $s=1$ and $s=0$ blocks of the above matrix are

$$\beta^0_{(1)} = \begin{pmatrix} 0 & \alpha & \beta \\ \alpha' & 0 & 0 \\ \beta' & 0 & 0 \end{pmatrix}, \qquad \beta^0_{(0)} = \begin{pmatrix} 0 & \gamma \\ \gamma' & 0 \end{pmatrix}. \tag{5.1}$$

The spin-1, unique-mass, conditions are that $\beta^0_{(0)}$ be nilpotent and

$$(\beta^0_{(1)})^3 = \beta^0_{(1)} \Rightarrow \alpha\alpha' + \beta\beta' = 1. \tag{5.2}$$

In the Duffin–Kemmer case, wherein the first row and column of $\beta^0_{(0)}$ don't exist, $\beta^0_{(0)}=0$ yielding, together with (5.2), the usual minimal equation $(\beta^0)^3=\beta^0$. However, with the inclusion of the extra scalar part—either γ or γ' or both being nonzero—$\beta^0_{(0)}$ is not null, but it can be made nilpotent,

$$(\beta^0_{(0)})^2=0, \tag{5.3}$$

by choosing $\gamma\gamma'=0$. With γ or γ' thus forced to vanish, the newly introduced scalar part of the wavefunction becomes a barnacle. With (5.2) and (5.3) as the minimal equations for $\beta^0_{(1)}$ and $\beta^0_{(0)}$, the minimal equation for β^0 is $(\beta^0)^4=(\beta^0)^2$. The degree of this equation being higher than $(2j_m+1)=3$, the Umezawa–Visconti theorem is seen to be violated. Further increase of the minimal degree by the introduction of more barnacles is readily achieved.

5.2. The Singh–Hagen equations

The arbitrary spin system of equations proposed by these authors (Singh and Hagen 1974a, b) provides a good illustration of some of the points raised above. As their equations for fermions have a relatively simple structure, we analyse these first before taking up the equations for bosons.

5.2.1. Equations for fermions of spin s. The Singh–Hagen equations for fermions (Singh and Hagen 1974b) are a generalisation of the familiar Rarita–Schwinger (1941) equation for spin-$\frac{3}{2}$ particles. The wavefunction transforms according to the reducible representation

$$\left(\frac{2s+1}{4},\frac{2s-1}{4}\right)+\left(\frac{2s-1}{4},\frac{2s+1}{4}\right)+\left(\frac{2s-1}{4},\frac{2s-3}{4}\right)+\left(\frac{2s-3}{4},\frac{2s-1}{4}\right)$$
$$+2\sum_{j=3/2}^{s-2}\left[\left(\frac{2j+1}{4},\frac{2j-1}{4}\right)+\left(\frac{2j-1}{4},\frac{2j+1}{4}\right)\right].$$

The equation describes particles of unique mass and half-integral spin s, which is also the highest spin (j_m) contained in ψ. In the following we shall take $s-\frac{1}{2}$ to be an odd integer. (This is necessary in order to be completely specific; the case of $s-\frac{1}{2}$ even is entirely parallel.) Then β^0 can be written in the form (2.4) by ordering the IR's in ψ as $D+\dot{D}$ where

$$D=\left(\frac{2s+1}{4},\frac{2s-1}{4}\right)+\left(\frac{2s-3}{4},\frac{2s-1}{4}\right)+2\left(\frac{2s-3}{4},\frac{2s-5}{4}\right)$$
$$+2\left(\frac{2s-7}{4},\frac{2s-5}{4}\right)+2\left(\frac{2s-7}{4},\frac{2s-9}{4}\right)+\ldots+2(0,\tfrac{1}{2})$$

and $\dot{D}$ contains the conjugate of the IR's in D. As the equations are parity invariant, A and B in β^0 of (2.4) are equal in the present case, and are square matrices. The spin-j block in β^0, namely $\beta^0_{(j)}$, will also have a similar form

$$\beta^0_{(j)}=\begin{pmatrix}0 & A_j\\ A_j & 0\end{pmatrix}\times I_j$$

which by a similarity transformation can be brought to $\begin{pmatrix}A_j & 0\\ 0 & -A_j\end{pmatrix}\times I_j$. Since the maximum spin s is contained in D only once, A_s is just a number (1×1 matrix) and hence it has to be equal to 1 in order to ensure that $(\beta^0_{(s)})^2 = 1$. For $j < s$, a count of the IR's in D which contain the spin j leads readily to the conclusion that A_j is a square matrix of dimension $(2s-2j)$. Clearly, the largest of these is $A_{1/2}$ with dimension $2s-1$, and its degree of nilpotency cannot exceed $2s-1$. The degree of nilpotency of the $\beta^0_{(1/2)}$ block is clearly the same as that of $A_{1/2}$ and we are able then to conclude that the degree of nilpotency of β^0 as a whole cannot exceed $2s-1$.

5.2.2. Equations for bosons of spin s. The familar equations for integer spin are second-order equations employing symmetric tensor representations of LG (Proca 1936, Takahashi and Palmer 1970, Shay and Good 1969, Velo 1972). To write equations for unique-mass particles of arbitrary integer spin s in first-order form, one has necessarily to introduce further tensors (both symmetric and non-symmetric). In their formulation Singh and Hagen (1974a) have constructed such a system of first-order equations by employing a wavefunction which transforms† as $D_1 + D_2$ where, for $s > 2$,

$$\begin{aligned} D_1 = {} & (\tfrac{1}{2}s, \tfrac{1}{2}s) + (\tfrac{1}{2}s-1, \tfrac{1}{2}s-1) + \ldots + (1,1) + (0,0) \\ & + [(\tfrac{1}{2}s-1, \tfrac{1}{2}s-2) + (\tfrac{1}{2}s-2, \tfrac{1}{2}s-1) + (\tfrac{1}{2}s-2, \tfrac{1}{2}s-2)] \\ & + [(\tfrac{1}{2}s-2, \tfrac{1}{2}s-3) + (\tfrac{1}{2}s-3, \tfrac{1}{2}s-2) + (\tfrac{1}{2}s-3, \tfrac{1}{2}s-3)] + \ldots \\ & + [(2,1) + (1,2) + (1,1)] + [(1,0) + (0,1)] + (0,0), \\ D_2 = {} & (\tfrac{1}{2}s-\tfrac{3}{2}, \tfrac{1}{2}s-\tfrac{3}{2}) + (\tfrac{1}{2}s-\tfrac{5}{2}, \tfrac{1}{2}s-\tfrac{5}{2}) + \ldots + (\tfrac{3}{2}, \tfrac{3}{2}) + (\tfrac{1}{2}, \tfrac{1}{2}) \\ & + [(\tfrac{1}{2}s+\tfrac{1}{2}, \tfrac{1}{2}s-\tfrac{1}{2}) + (\tfrac{1}{2}s-\tfrac{1}{2}, \tfrac{1}{2}s+\tfrac{1}{2}) + (\tfrac{1}{2}s-\tfrac{1}{2}, \tfrac{1}{2}s-\tfrac{1}{2})] \\ & + [(\tfrac{1}{2}s-\tfrac{1}{2}, \tfrac{1}{2}s-\tfrac{3}{2}) + (\tfrac{1}{2}s-\tfrac{3}{2}, \tfrac{1}{2}s-\tfrac{1}{2})] + [(\tfrac{1}{2}s-\tfrac{3}{2}, \tfrac{1}{2}s-\tfrac{5}{2}) \\ & + (\tfrac{1}{2}s-\tfrac{5}{2}, \tfrac{1}{2}s-\tfrac{3}{2}) + (\tfrac{1}{2}s-\tfrac{5}{2}, \tfrac{1}{2}s-\tfrac{5}{2})] + \ldots + [(\tfrac{3}{2}, \tfrac{1}{2}) + (\tfrac{1}{2}, \tfrac{3}{2}) + (\tfrac{1}{2}, \tfrac{1}{2})] + (\tfrac{1}{2}, \tfrac{1}{2}). \end{aligned}$$

The β^0 of the theory corresponding to the above ordering of the representations will have the form (2.4) and, correspondingly, the spin-j block will automatically have the similar form

$$\beta^0_{(j)} = \begin{pmatrix} 0 & A_j \\ B_j & 0 \end{pmatrix} \times I_j.$$

Unlike in the fermion case, the matrices A_j, B_j are now rectangular, since the number of IR's involved in D_1 and D_2 are not equal.

To determine the dimension of A_j and B_j, let us start with $j_m = s$ (the maximum spin in ψ) which is also the spin of the particle described by the equation. It is easily seen that one IR, $(\frac{1}{2}s, \frac{1}{2}s)$, in D_1 and two IR's, $(\frac{1}{2}s+\frac{1}{2}, \frac{1}{2}s-\frac{1}{2})$ and $(\frac{1}{2}s-\frac{1}{2}, \frac{1}{2}s+\frac{1}{2})$, in D_2 contain s and therefore A_s is a 1×2 matrix and B_s is a 2×1 matrix. Thus $\begin{pmatrix} 0 & A_s \\ B_s & 0 \end{pmatrix}$ is a 3×3 matrix. Since it must give rise to the eigenvalues ± 1 once each (and the other eigenvalue must be zero) its minimal equation has to be of the form $(\beta^0_{(s)})^3 = \beta^0_{(s)}$. For the next lower

† We shall assume s to be an even integer in order to make the groupings of the IR's into D_1 and D_2 completely specific. The case of odd s is entirely parallel.

For $s < 2$ the equations can be formulated with a smaller number of IR's than indicated by the general scheme and so they are treated as a special case by these authors (see also Hagen 1971, Chang 1967, Schwinger 1963).

spin, $(s-1)$, A_{s-1} is a 1×3 matrix (B_{s-1} is 3×1) and so it might appear that the maximum degree of nilpotency of $\begin{pmatrix} 0 & A_{s-1} \\ B_{s-1} & 0 \end{pmatrix}$ is 4. But it cannot exceed 3 for the following reason. The rank of A_{s-1} is 1 and so is that of B_{s-1}. Therefore, the rank r_{s-1} of the matrix $\begin{pmatrix} 0 & A_{s-1} \\ B_{s-1} & 0 \end{pmatrix}$ is 2. Hence the degree of nilpotency (which is $r_{s-1}+1$ at most) is 3. A similar situation arises for other lower spins as well and the degree of nilpotency of any of the lower spin blocks is much less than its dimension. Let us demonstrate this for the spin-1 skeleton sub-block which has the highest dimension of all the spin blocks. A_1 is a $(2s-4)\times(2s-1)$ matrix and B_1 is a $(2s-1)\times(2s-4)$ matrix. Based on the argument given above for the $(s-1)$ block, we conclude that the rank of $\begin{pmatrix} 0 & A_1 \\ B_1 & 0 \end{pmatrix}$ cannot exceed $2(2s-4)$. Hence the degree of nilpotency l_1 is $<(4s-7)$. If the value of l_1 were indeed $(4s-7)$ the minimal degree of β^0 would be $4s-5$, which for large s would be much greater than $2s+1$—the Umezawa–Visconti upper limit. It turns out however, as shown below, that the value of l_1 does not exceed $(2s-1)$ and therefore the minimal degree of β^0 will be $(\beta^0)^{l+2}=(\beta^0)^l$ with $l\leqslant(2s-1)$. To demonstrate this, let us define, following Singh and Hagen, tensors $\phi^{(k)}(k=0,\ldots,s)$ which transform according to the IR $(\frac{1}{2}k,\frac{1}{2}k)$ and $H^{(p)}$ $(p=2,\ldots,s-3,s-1)$ with the transformation property $\{[(p+1)/2,(p-1)/2]+[(p-1)/2,(p+1)/2]+[(p-1)/2,(p-1)/2)]\}$. $H^{(s-2)}$ transforms as $[(s-1)/2,(s-3)/2]+[(s-3)/2,(s-1)/2)]$, $H^{(1)}$ as $(0,1)+(1,0)$, $H^{(0)}$ as $(\frac{1}{2},\frac{1}{2})$ and H as $(0,0)$. From the equations of motion one can readily see that the form of the A_1 matrix will be as follows:

$$
\begin{array}{c|ccccc|cccccc}
 & \phi^{(s-3)} & \phi^{(s-5)} & \cdots & \phi^{(3)} & \phi^{(1)} & H^{(s)} & H^{(s-2)} & H^{(s-4)} & \cdots & H^{(2)} & H^{(0)} \\
\hline
\phi^{(s)} & & & & & & + & & & & & \\
\phi^{(s-2)} & \times & & & & & & + & & & & \\
\phi^{(s-4)} & \times & \times & & & & & & + & & & \\
\vdots & & & & & & & & & & & \\
\phi^{(2)} & & & & \times & \times & & & & & + & \\
\hline
H^{(s-3)} & \surd & & & & & & & & & & \\
H^{(s-5)} & & \surd & & & & & & & & & \\
\vdots & & & & & & & & & & & \\
H^{(3)} & & & & \surd & & & & & & & \\
H^{(1)} & & & & & \surd & & & & & &
\end{array}
$$

(The crosses indicate single numbers while the $+$ symbols stand for 1×3 matrices (except for the $H^{(s-2)}$ column where it is 1×2), the tick marks ($\surd$) are 3×1 matrices and blanks are zero.) By elementary transformations one can easily convince oneself that the rank of the above matrix does not exceed $(s-1)$. By an exactly similar construction one can verify that the rank of B_1 also does not exceed $(s-1)$. Therefore the rank r_1 of

$\begin{pmatrix} 0 & A_1 \\ B_1 & 0 \end{pmatrix}$ cannot exceed $(2s-2)$, and hence the minimal degree of β^0 will be subject to the bound stated above.

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